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Convection of a passive scalar by a quasi-uniform random straining field

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The stretching of line elements, surface elements and wave vectors by a random, isotropic, solenoidal velocity field in D dimensions is studied. The rates of growth of line elements and $(D-1)$-dimensional surface elements are found to be equal if the statistics are invariant to velocity reversal. The analysis is applied to convection of a sparse distribution of sheets of passive scalar in a random straining field whose correlation scale is large compared with the sheet size. This is Batchelor's (1959) k^{-1} spectral regime. Some exact analytical solutions are found when the velocity field varies rapidly in time. These include the dissipation spectrum and a joint probability distribution that describes the simultaneous effect of stretching and molecular diffusivity κ on the amplitude profile of a sheet. The latter leads to probability distributions of the scalar field and its space derivatives. For a growing k^{-1} range at zero κ, these derivatives have essentially lognormal statistics. In the steady-state k^{-1} regime at $\kappa > 0$, intermittencies measured by moment ratios are much smaller than for lognormal statistics, and they increase less rapidly with the order of the derivative than in the $\kappa = 0$ case. The $\kappa > 0$ distributions have singularities at zero amplitude, due to a background of highly diffused sheets. The results do not depend strongly on D. But as $D \to \infty$, temporal fluctuations in the stretching rates become negligible and Batchelor's (1959) constant-strain dissipation spectrum is recovered.

1. Introduction

In recent years there has been a substantial theoretical and experimental effort exerted in the study of small-scale statistics and intermittency effects in turbulent flows. In a previous paper (Kraichnan 1974), we have attempted to analyse some of the theoretical ideas about the statistics of the inertial range and to point out how wide a range of possibilities remains open until some detailed analysis based on real use of the Navier–Stokes equation can be brought to bear. The convection of passive-scalar blobs by a velocity field whose shear is almost constant over a blob is an attractive problem in this context. In introducing the problem, Batchelor (1959) obtained exact results (a rare thing in turbulence theory) for the spectrum power law at wavenumbers where molecular diffusivity is unimportant and for the dissipation spectrum, the latter result being restricted to the case of strains which are very persistent in time. Kraichnan (1968) found the dissipation spectrum for the opposite case of a velocity field with a very short correlation time. In the present paper, the exact results for a rapidly varying velocity field are extended to a variety of higher statistics, including intermittency of spatial derivatives of the scalar field, for the case where the initial, or input, scalar field consists of a collection of sparsely distributed little sheets, of arbitrary amplitude profile. The analysis is carried out for an incompressible velocity field in a space of arbitrary dimension D.

Because exact results can be obtained, this problem is a good testing ground for general ideas, about the build-up of intermittency through repeated random straining or cascade, which have been applied to small-scale turbulence statistics. In particular, we are able to examine precisely the effect of molecular dissipation on essentially lognormal statistics which are built up in the passage of the scalar blobs through the k^{-1} range. Because the scalar dynamics are linear, it is possible to sort out effects of velocity-field statistics from those of the input scalar statistics.

Treatments analogous to the present one, although with less possibility of exact results, can be carried out for blobs of passive scalar cascading through a $-\frac{5}{3}$ spectral regime, for inertial-range cascade in three-dimensional Navier–Stokes turbulence with an initial condition of sparsely distributed thin vortex rings, and for enstrophy cascade in two-dimensional turbulence. The analogy with the last-named problem is particularly close. Because of these connexions, the present analysis is carried out in considerable detail, and with fairly substantial formal trappings.

The work is carried out in a space of general dimension D in order to see whether there are qualitative differences in behaviour for different D and, in particular, to examine the asymptotic behaviour as $D \to \infty$. The desirability of doing this was pointed out to me by Dr M. Nelkin, on the basis of possible analogies with the dependence of properties on dimensionality in critical-point phenomena. Our results are perhaps suggestive. For $D \to \infty$, temporal fluctuations in the stretching of scalar blobs become negligible, and Batchelor's original dissipation spectrum based on constant strain is recovered.

Section 7 of the paper contains a detailed summary of the analysis. It should probably be looked over before starting the body of the paper.

2. Stretching of lines, wave vectors and surfaces

An infinitesimal line element r_i moving with the fluid velocity $u_i(\mathbf{x}, t)$ stretches according to

$$dr_i/dt = a_{ij}(\mathbf{x}, t)\, r_j, \quad a_{ij}(\mathbf{x}, t) = \partial u_i(\mathbf{x}, t)/\partial x_j. \tag{2.1}$$

Hence

$$\mathbf{r}(t) = \mathbf{W}(t, \mathbf{x}; t', \mathbf{x}') \,.\, \mathbf{r}(t'), \tag{2.2}$$

with

$$\mathbf{W}(t, \mathbf{x}; t', \mathbf{x}') = \exp_{\leftarrow}\left[\int_{t'}^{t} \mathbf{a}(\mathbf{y}, s)\, ds\right] \tag{2.3}$$

(Cocke 1969). In (2.3), $\exp_{\leftarrow}$ means that all the matrix factors $\mathbf{a}$ in the expansion of the exponential are arranged from right to left in order of increasing time argument; $\mathbf{y} = \mathbf{y}(s)$ is the position of the line element in the fluid at time s, with $\mathbf{y}(t') = \mathbf{x}'$ and $\mathbf{y}(t) = \mathbf{x}$.

Consider a field $\phi(\mathbf{x}, t)$ which moves with the fluid according to

$$\partial\phi/\partial t+\mathbf{u}.\nabla\phi = 0. \tag{2.4}$$

At time t, suppose it has locally the form $\phi(\mathbf{x}, t) = \exp(i\mathbf{k}.\mathbf{x})$, where k^{-1} is infinitesimal. Then in an infinitesimal neighbourhood of $\mathbf{x}$

$$\begin{aligned}\phi(\mathbf{x}+\mathbf{r}, t+dt) &= \exp\{i\mathbf{k}.[\mathbf{x}+\mathbf{r}-\mathbf{u}(\mathbf{x},t)\,dt-\mathbf{a}(\mathbf{x},t).\mathbf{r}\,dt]\}\\ &= \exp\{i\mathbf{k}.[\mathbf{x}-\mathbf{u}(\mathbf{x},t)\,dt]\}\exp\{i[\mathbf{k}-\mathbf{k}.\mathbf{a}(\mathbf{x},t)\,dt].\mathbf{r}\}.\end{aligned}$$

Hence the local wave vector $\mathbf{k}(t)$ obeys

$$dk_i/dt = -k_j a_{ji}(\mathbf{x}, t), \tag{2.5}$$

and

$$\mathbf{k}(t) = \mathbf{k}(t').\mathbf{W}^{-1}(t, \mathbf{x}; t', \mathbf{x}'), \tag{2.6}$$

with

$$\mathbf{W}^{-1}(t, \mathbf{x}; t', \mathbf{x}') = \exp_{\rightarrow}\left[-\int_{t'}^{t}\mathbf{a}(\mathbf{y}, s)\,ds\right]. \tag{2.7}$$

Here $\exp_{\rightarrow}$ denotes time-ordering from left to right and $\mathbf{W}.\mathbf{W}^{-1} = \mathbf{W}^{-1}.\mathbf{W} = \mathbf{I}$, the unit matrix. It follows from (2.2) and (2.6) that

$$\mathbf{k}(t).\mathbf{r}(t) = \mathbf{k}(t').\mathbf{r}(t'), \tag{2.8}$$

if $\mathbf{r}$ and $\mathbf{k}$ are any line element and local wavenumber which are carried along in the same fluid element.

The normal vector terminating on neighbouring surfaces of zero phase of ϕ in the preceding analysis is $\pi\mathbf{k}/k^2$. Since $\mathbf{k}$ is arbitrary, it follows that the normal vector $\mathbf{b}$ connecting any two infinitesimally separated parallel surface elements moving with the fluid has a reciprocal vector $\mathbf{k} = \mathbf{b}/b^2$ which obeys (2.5) and (2.6). Substitution into (2.5) gives for $\mathbf{b}$ itself the equation

$$db_i/dt = -b_j a_{ji}+2b_i b_n b_j b^{-2}a_{jn}. \tag{2.9}$$

Now consider an ensemble of $\mathbf{r}(t')$ and $\mathbf{k}(t')$ which are statistically independent of the velocity field and statistically isotropic at all $\mathbf{x}'$:

$$\langle\mathbf{r}(t')\,\mathbf{r}(t')\rangle/|\mathbf{r}(t')|^2 = \langle\mathbf{k}(t')\,\mathbf{k}(t')\rangle/\langle|\mathbf{k}(t')|^2\rangle = D^{-1}\mathbf{I}, \tag{2.10}$$

where D is the number of space dimensions. It follows directly from (2.2), (2.6), (2.10) and the independence assumption that

$$\langle r^2(t)\rangle/\langle r^2(t')\rangle = D^{-1}\langle\mathrm{tr}\,[\mathbf{W}(t, \mathbf{x}; t', \mathbf{x}').\tilde{\mathbf{W}}(t, \mathbf{x}; t', \mathbf{x}')]\rangle, \tag{2.11a}$$

$$\langle k^2(t)\rangle/\langle k^2(t')\rangle = D^{-1}\langle\mathrm{tr}\,[\tilde{\mathbf{W}}^{-1}(t, \mathbf{x}; t', \mathbf{x}').\mathbf{W}^{-1}(t, \mathbf{x}; t', \mathbf{x}')]\rangle. \tag{2.11b}$$

Here $r^2(t) = |\mathbf{r}(t)|^2$, tr denotes trace, $\tilde{W}_{ij} = W_{ji}$ and $(\tilde{W}^{-1})_{ij} = (W^{-1})_{ji}$. $\tilde{\mathbf{W}}^{-1}$ is also the inverse of $\tilde{\mathbf{W}}$, satisfying $\tilde{\mathbf{W}}.\tilde{\mathbf{W}}^{-1} = \tilde{\mathbf{W}}^{-1}.\tilde{\mathbf{W}} = \mathbf{I}$.

Several things should be noted about the derivation of (2.11). We have made no restriction on D ($D \geqslant 1$) nor have we assumed incompressible flow. If the flow is compressible, (2.4) is not the usual equation for a passive scalar; we use it purely as a device. No assumption has been made about the velocity-field statistics, nor about the probability distribution of the initial positions $\mathbf{x}'$ of the line elements and local wave vectors.

Now suppose that (i) the flow is incompressible (which requires $D > 1$) and (ii) all averages over the velocity field are invariant to the velocity reversal $\mathbf{u}(\mathbf{x}, t'+s) \to -\mathbf{u}(\mathbf{x}, t-s) \quad (0 \leqslant s \leqslant t-t')$. It follows from (2.11) that

$$[\langle r^2(t)\rangle/\langle r^2(t')\rangle]_{\mathbf{x}'} = [\langle k^2(t)\rangle/\langle k^2(t')\rangle]_{\mathbf{x}'}, \tag{2.12}$$

where $[\;]_{\mathbf{x}'}$ denotes a space average over an initial uniform distribution of the starting positions $\mathbf{x}'$ over the whole volume of the fluid. The proof is short. A line element (or local wave vector) which originally follows a trajectory from $(\mathbf{x}', t')$ to $(\mathbf{x}, t)$ would, under the reversal, retrace the trajectory to $(\mathbf{x}', t)$, if initially placed at $(\mathbf{x}, t')$. In doing so, it would suffer just the reverse of the original stretching and rotation. This implies that, under the velocity reversal

$$\mathbf{W}(t, \mathbf{x}; t', \mathbf{x}') \to \mathbf{W}^{-1}(t, \mathbf{x}'; t', \mathbf{x}), \tag{2.13}$$

a result which is also easy to see from (2.3) and (2.7). The use of (2.13) in (2.11) gives

$$[\langle k^2(t)\rangle/\langle k^2(t')\rangle]_{\mathbf{x}'} = [\langle \mathrm{tr}\,[\widetilde{\mathbf{W}}(t, \mathbf{x}'; t', \mathbf{x}) . \mathbf{W}(t, \mathbf{x}'; t', \mathbf{x})]\rangle]_{\mathbf{x}'}, \tag{2.14}$$

under the assumption of reversal invariance. Since the flow is incompressible, the probability measure of the element positions is conserved, so that a uniform distribution of positions $\mathbf{x}'$ implies a uniform distribution of positions $\mathbf{x}$, and vice versa. Consequently, the value of the right-hand side of (2.14) is unchanged if $[\;]_{\mathbf{x}'}$ is replaced by $[\;]_{\mathbf{x}}$. The result is identical with the average $[\;]_{\mathbf{x}'}$ of (2.11a), yielding (2.12). (Note that in the space averaging $\mathbf{x}$ and $\mathbf{x}'$ are integration variables, only one of which is independent.) Equation (2.12) also follows if the velocity statistics lack the time-reversal invariance (ii) but are invariant to reflexion about a centre of spatial symmetry, a particular case being reflexion-invariant homogeneous turbulence.

Now let $\mathbf{k}$ be the reciprocal of the thickness vector of an infinitesimal thin slab of fluid whose $(D-1)$-dimensional surface area is $S(t)$. Incompressibility implies that $k(t)/S(t)$ is independent of t. Thus (2.12) yields

$$[\langle S^2(t)\rangle/\langle S^2(t')\rangle]_{\mathbf{x}'} = [\langle r^2(t)\rangle/\langle r^2(t')\rangle]_{\mathbf{x}'}, \tag{2.15}$$

which says that the mean-square stretching of a randomly oriented and placed surface element is equal to the mean-square stretching of a random line element. For $D \geqslant 3$, equation (2.15) is perhaps surprising. At first sight, it might be expected that each edge vector of a surface element should suffer effectively independent stretching, so that the net rate of stretching of the surface area should have a greater mean-square value than that of a single edge. However, the stretching of the different edges is not really independent, and it is plausible that an initially square surface element should be stretched out into a parallelogram whose acute angle becomes typically smaller and smaller as stretching proceeds. This would have the effect of reducing the rate of increase of the area.

The evolution of the surface element $S_{ij}(t) = r_i^{(1)}(t)\, r_j^{(2)}(t)$, where $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$ are the edge vectors ($D = 3$), can be investigated directly with the equation of motion

$$dS_{ij}(t) = a_{in}(\mathbf{x}, t)\, S_{nj}(t) + a_{jn}(\mathbf{x}, t)\, S_{in}(t), \tag{2.16}$$

which follows from (2.1). The squared area is given by

$$S^2(t) = S_{ij}(t)\, S_{ij}(t) - (S_{ii})^2. \tag{2.17}$$

Reversal invariance is an unphysical assumption for Navier–Stokes turbulence. It means, for example, that the energy transfer function vanishes. However, a Gaussianly distributed velocity field, produced, say, by random stirring at low Reynolds number, could display invariance to $\mathbf{u}$ reversal. In the limit where the random velocity field has a correlation time very short compared with the eddy circulation time (fast stirring), some explicit results can be obtained for the logarithmic rate of stretching of line and surface elements.

Let $\mathbf{u}(\mathbf{x}, t)$ be a statistically isotropic and homogeneous field which is a white-noise process in time. It may be constructed by the following limiting process. Take $u_i(\mathbf{x}, t)$ to be piecewise constant on intervals Δt, with $u_i(\mathbf{x}, t) \propto (\Delta t)^{-\frac{1}{2}}$ and statistically independent for different intervals. Then take $\Delta t \to 0$. To order Δt, equation (2.1) gives

$$r_i(t+\Delta t) = r_i(t) + a_{ij}(t)\, r_j(t)\, \Delta t + \tfrac{1}{2} a_{ij}(t)\, a_{jn}(t)\, r_n(t)\, (\Delta t)^2, \tag{2.18}$$

where $a_{ij}(t)$ is the constant value in the interval $(t, t+\Delta t)$. No account is taken in (2.18) of the change in trajectory $\mathbf{y}(t)$ during the interval, because this turns out to give a vanishing effect in the limit $\Delta t \to 0$.† After some index manipulation, the scalar product of (2.18) with itself gives

$$\Delta r^2 = 2a_{ij} r_i r_j \Delta t + a_{ij}(a_{in} + a_{ni})\, r_j r_n (\Delta t)^2, \tag{2.19}$$

to order Δt, where $\Delta r^2 = r^2(t+\Delta t) - r^2(t)$.

Now suppose that the initial distribution of $\mathbf{r}$ is isotropic. It stays isotropic, since the velocity statistics are isotropic, and moreover, $a_{ij}(t)$ is statistically isotropic and statistically independent of $\mathbf{r}(t)$, by the assumptions on the velocity field. It follows that the statistics of $r^2(t+\Delta t)$ are unchanged if (2.19) is replaced by

$$\Delta r^2 = 2a_{ij} n_i n_j (\Delta t)\, r^2 + a_{ij}(a_{in} + a_{ni})\, n_j n_n (\Delta t)^2\, r^2, \tag{2.20}$$

where $\mathbf{n}$ is a statistically isotropic unit vector, independent of a_{ij}.

The coefficient of Δt on the right-hand side of (2.20) has zero mean and is $O[(\Delta t)^{-\frac{1}{2}}]$. Thus the first term on the right-hand side gives a contribution $O[(\Delta t)^{\frac{1}{2}}]$ to Δr^2. In the limit $\Delta t \to 0$, this results in finite changes in r^2 in finite times, in accord with the '$N^{\frac{1}{2}}$ law'. The second term on the right-hand side is quadratic in $\mathbf{a}$. Consequently, it has a mean which is $O(\Delta t)$ and a fluctuating part also $O(\Delta t)$. In the limit $\Delta t \to 0$, only the mean can give a finite change in r^2 in a finite time; the fluctuating part of the second term is negligible in the limit.

If (2.20) is substituted into

$$\Delta r = (r^2 + \Delta r^2)^{\frac{1}{2}} - r = [(1 + \Delta r^2/r^2)^{\frac{1}{2}} - 1]\, r$$

† To order Δt, the change in trajectory adds the term $\frac{1}{2}(\partial a_{ij}/\partial x_n)\, u_n r_j (\Delta t)^2$ to (2.18). In accord with the argument following (2.20), only the mean of this term contributes to the growth of r^2 when $\Delta t \to 0$, and the mean vanishes because of homogeneity, incompressibility and the independence of $\mathbf{r}(t)$ and $\mathbf{u}(t)$. That independence follows from the independence of $\mathbf{u}$ in $(t, t+\Delta t)$ of its previous values.

and the radical is expanded, the result to order Δt is

$$\Delta r = a_{ij} n_i n_j (\Delta t) r + \tfrac{1}{2}[a_{ij}(a_{in} + a_{ni}) n_j n_n - a_{ij} a_{rs} n_i n_j n_r n_s] (\Delta t)^2 r. \tag{2.21}$$

Now try to represent the evolution of $r(t)$ by the scalar equation

$$dr/dt = a(t) r, \tag{2.22}$$

where $a(t)$ is a white-noise process. If $a(t)$ is taken to be piecewise constant, as in the analysis above, the result to order Δt is

$$\Delta r = [a\Delta t + \tfrac{1}{2}a^2(\Delta t)^2] r. \tag{2.23}$$

Write $a = \langle a \rangle + a'$, where $\langle a \rangle$ and a' are the mean and fluctuating parts. Equations (2.21) and (2.23) are the same, to order Δt in mean parts and order $(\Delta t)^{\frac{1}{2}}$ in fluctuating parts, if

$$a' = a_{ij} n_i n_j, \tag{2.24}$$

$$\begin{aligned} \langle a \rangle &= \tfrac{1}{2}\{\langle [\;] \rangle - \tfrac{1}{2}\langle a'^2 \rangle\} \Delta t \\ &= \langle [\tfrac{1}{2}a_{ij}(a_{in} + a_{ni}) n_j n_n - a_{ij} a_{rs} n_i n_j n_r n_s] \rangle \Delta t, \end{aligned} \tag{2.25}$$

where [ ] denotes the square-bracketed expansion in (2.21).

In the limit $\Delta t \to 0$, $\langle a_{ij} a_{rs} \rangle \Delta t \to 2A_{ijrs}$, where

$$A_{ijrs}(\mathbf{x}, t) = \int_{-\infty}^{t} \langle a_{ij}(\mathbf{x}, t) a_{rs}(\mathbf{x}, s) \rangle ds, \tag{2.26}$$

and the factor 2 occurs because the continuous t in (2.26) falls with equal probability anywhere in the interval Δt. The isotropic average formulae

$$\langle n_i n_j \rangle = D^{-1} \delta_{ij}, \quad \langle n_i n_j n_r n_s \rangle = [D(D+2)]^{-1} (\delta_{ij}\delta_{rs} + \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})$$

used in (2.24) and (2.25), together with the independence of **a** and **n**, yield

$$\langle a(t) \rangle = A(t)/(D+2), \quad c(t) \equiv \int_{-\infty}^{t} \langle a'(t) a'(s) \rangle ds = A(t)/[D(D+2)], \tag{2.27}$$

where $A(t) = A_{ijij}(t)$. In writing (2.27) we have assumed incompressibility, which implies (via homogeneity and a partial integration) $A_{iijj} = A_{ijji} = 0$.

If the preceding analysis is carried out starting with (2.5) instead of (2.1), the only change is $a_{ij} \to a_{ji}$ throughout, and the result is

$$dk/dt = a(t) k, \tag{2.28}$$

with $a(t)$ again a white-noise process specified by (2.27) and the property that $a(t)$ and $a(t')$ are statistically independent for $t \neq t'$. This corroborates (2.12).

According to (2.22), $q(t) = \ln[r(t)/r(0)]$ satisfies

$$q(t) = \int_0^t a(s) ds. \tag{2.29}$$

Our conditions on $a(t)$ imply that $q(t)$ is normally distributed, by the central limit theorem, so that its statistical distribution is fully specified by the mean and covariance

$$\langle q(t) \rangle = \int_0^t \langle a(s) \rangle ds, \quad \langle q'(t) q'(t') \rangle = 2\int_0^{t'} c(s) ds \quad (t \geqslant t'). \tag{2.30}$$

The univariate probability distribution is

$$Q(q) = [4\pi ct]^{-\frac{1}{2}} \exp[-(q-\langle a\rangle t)^2/4ct], \tag{2.31}$$

in the special case where $a(t)$ is a stationary process.

Suppose now that the velocity field is homogeneous and isotropic but has finite correlation times. The statistical symmetry implies that the probability distribution of the length $r(t)$ of a randomly placed line element of initial length $r(0)$ depends only on $r(0)$; that is

$$r(t) = w(t)\, r(0), \tag{2.32}$$

where the random process $w(t)$ is a functional of the velocity field. Differentiating, one obtains (2.22) with $a(t) = d(\ln w)/dt$. Now $a(t)$ is not a white-noise process, but as before, it has both a mean and a fluctuating part. Cocke (1969) shows in general that $\langle \ln w\rangle > 0$, in accord with our result for the white-noise case.

It is important to note that the non-zero mean $\langle a(t)\rangle$ arises not because there are mean shears (there are not in the cases described) but as a consequence of the fact that $D > 1$. It is this fact that results in non-zero mean logarithmic rates of increase for $|\mathbf{r}(t)|$ even though the tensor $\mathbf{a}(t)$ has zero mean.† Note that, in the incompressible white-noise case (2.27),

$$\langle a(t)\rangle / c(t) = D, \tag{2.33}$$

so that the ratio rises with dimensionality D. In the more realistic case where $a(t)$ and the characteristic frequencies of $a(t)$ are both of the order of typical shears in the fluid, we conjecture that (2.33) is still approximately valid.

3. The k^{-1} regime at zero diffusivity

The analysis of §2 applies directly to the stretching of blobs of a passively convected scalar field at zero molecular diffusivity, provided that the spatial scale of variation of the straining field is large compared with the blob sizes. Batchelor (1959) was the first to treat this regime. He found that the scalar spectrum $F(k)$ for wavenumbers k large compared with those contributing to the straining field should have the steady-state form

$$F(k) = \chi\gamma k^{-1}, \tag{3.1}$$

where χ is the rate at which scalar variance is transferred into the k^{-1} range from lower wavenumbers and γ is an effective rate-of-strain parameter of order $(\epsilon/\nu)^{\frac{1}{2}}$, with ϵ and ν equal to the rate of energy dissipation by the turbulence per unit mass and the kinematic viscosity, respectively.

Consider an initial state in which the scalar field consists of small flat sheets or ribbons whose thickness b is small compared with the other dimensions. Let the sheets be homogeneously and isotropically distributed and let the initial thickness

† A rather delicate distinction arises here. If $d\langle \ln[r(t)/r(0)]\rangle/dt > 0$, then $\langle a(t)\rangle > 0$. But, in general, $d[\ln\langle r(t)/r(0)\rangle]/dt > 0$ even if $\langle a(t)\rangle = 0$. If the velocity field is statistically stationary and its correlations extend only over finite times, Cocke's result $\langle \ln w\rangle > 0$ implies that $\langle a(t)\rangle > 0$.

$b(0)$ be the same for all sheets. For the moment we leave unspecified the variation of scalar amplitude across the sheets. Starting at $t = 0$, let the sheets be convected by a stationary, homogeneous isotropic turbulent velocity field which is statistically independent of the initial scalar field. Let the sheets be small enough that the strain field varies negligibly over each sheet during the entire convection process. Then, by the argument yielding (2.32) and its consequences, the reciprocal vector to $\mathbf{b}(t)$ obeys an equation of the form of (2.28), so that b satisfies

$$db/dt = -a(t)\,b. \tag{3.2}$$

If $q(t)$ is now redefined as $-\ln[b(t)/b(0)]$, it satisfies (2.29).

If the straining field is a white-noise process, as discussed in §2, the probability distribution is given by (2.31) and (2.27). The probability distribution of b itself is

$$P(b) = Q(q)|dq/db| = b^{-1}Q[-\ln(b/b_0)], \tag{3.3}$$

with $b_0 = b(0)$. When the straining field has a finite correlation time, q as given by (2.29) is asymptotically normal, with distribution of the form (2.31), provided that $a(t)$ satisfies sufficiently strong conditions of independence for time differences long compared with the correlation time (Lumley 1972; Rosenblatt 1972). We shall discuss these conditions, and the nature of the asymptotic approach to normality for large t, later on. It should be noted that (2.27) holds only for the white-noise case.

Equations (3.3) and (2.31) imply that $P(b)$ behaves like b^{-1} over the range

$$|q - \langle a\rangle t| \ll (ct)^{\frac{1}{2}}, \tag{3.4}$$

a range whose logarithmic length grows like $t^{\frac{1}{2}}$ and whose centre propagates to smaller b values at the logarithmic rate $\langle a\rangle$. There is a corresponding k^{-1} range in the scalar spectrum function $F(k)$. To see this, suppose that $F(k)$ has the initial form

$$F(k) = \Psi b_0 f(kb_0), \tag{3.5}$$

where Ψ is the scalar variance and f, which satisfies

$$\int_0^\infty f(x)\,dx = 1,$$

is related to the Fourier transform of the amplitude profile across the sheets. This form for $F(k)$ is assumed to hold for k^{-1} small compared with the edge lengths of the sheets.

Under our assumption that straining is negligibly non-uniform over each sheet, the shape of the amplitude profile is unchanged during convection, and it follows that the spectrum at later times is

$$F(k) = \Psi\int_0^\infty bf(kb)\,P(b)\,db. \tag{3.6}$$

Suppose that $f(x)$ is monotonically decreasing, with half-width of order one. Examples would be $f(x) = 1,\ 0$ for $x \lessgtr 1$, or $f(x) = (2/\pi)^{\frac{1}{2}}\exp(-\frac{1}{2}x^2)$. Then

$F(k)$ decreases monotonically from a maximum at $F(0)$. If (3.3) and (2.31) are used in (3.6), the result is that $F(k)$ has a half-width measured by

$$\ln (b_0 k) = \langle a \rangle t$$

and displays a k^{-1} range given by

$$F(k) \approx \Psi[4\pi ct]^{-\frac{1}{2}} k^{-1} \quad [|\ln (b_0 k) - \langle a \rangle t| \ll (ct)^{\frac{1}{2}}]. \tag{3.7}$$

The rough behaviour for k beyond the k^{-1} range is given by replacing $f(x)$ with $\delta(x-1)$. Then (3.6) yields

$$F(k) \sim k^{-2} P(k^{-1}) \quad [\ln (b_0 k) \gg \langle a \rangle t + (ct)^{\frac{1}{2}}], \tag{3.8}$$

to logarithmic approximation. This falls off more rapidly than any power of k if (3.3) and (2.31) are used.

A similar k^{-1} spectrum arises also under more general conditions than the present lognormal, or asymptotically lognormal, statistics. All that is needed is that

$$\int_0^t a(s)\, ds$$

exhibit a probability distribution that falls off smoothly and rapidly enough from a maximum at the mean value, and has a half-width that continually increases with time. Then (3.3) and (3.5) imply a steadily lengthening range of b^{-1} and k^{-1} behaviour.

Moreover, if a b^{-1} range is set up as an initial condition, it persists for a time, regardless of the $a(t)$ statistics. Write $P(b)$ as $\langle \hat{P}(b) \rangle$, where the angular brackets denote averaging over $a(t)$ and, by (3.2), $\hat{P}(b)$ obeys the continuity equation in phase space,

$$\partial \hat{P}(b)/\partial t = a(t)\, \partial [b \hat{P}(b)]/\partial b. \tag{3.9}$$

Clearly $\hat{P}(b)$ does not change with time in a region where it varies like b^{-1}. If $\hat{P}(b)$ varies initially like b^{-1} in a range of b, then $P(b)$ is constant in time over that range until there has accumulated an appreciable statistical weight of stretchings which connect b values within the range to values outside the initial b^{-1} range.

Now suppose that, starting at $t = 0$, randomly placed and oriented sheets of thickness b_0 are added at a statistically steady rate R. Let the associated rate of input of scalar variance be χ, so that, if there were no convection, the spectrum at time t would be given by (3.5) with $\Psi = \chi t$. For a white-noise $a(t)$, the probability distribution $Q(q)$, normalized with Rt, is the integral of (2.31):

$$Q(q) = R \int_0^t (4\pi cs)^{-\frac{1}{2}} \exp [-(q - \langle a \rangle s)^2/4cs]\, ds. \tag{3.10}$$

This yields a b^{-1} range specified by

$$P(b) \approx R(\langle a \rangle b)^{-1} \quad [\langle a \rangle t \gg (ct)^{\frac{1}{2}}, \quad 0 \ll \ln (b_0/b) \ll \langle a \rangle t], \tag{3.11}$$

since, under the stated inequalities, the integral in (3.10) is $\approx \langle a \rangle^{-1}$.† By (3.6), the associated k^{-1} range is

$$F(k) \approx \chi(\langle a \rangle k)^{-1} \quad (0 \ll \ln (kb_0) \ll \langle a \rangle t). \tag{3.12}$$

† In effect, the integral may be replaced by $\int_0^t \delta(q - \langle a \rangle s)\, ds$.

In three dimensions, this result, with $\langle a\rangle$ given by (2.27), has previously been obtained by a different analysis, involving directly the scalar field equation (Kraichnan 1968).

Equation (3.12) differs from Batchelor's original result (3.1) only in that $\langle a\rangle$ does not really correspond to a persistent least-rate of strain $-\gamma$ as invoked by Batchelor (1959). The constant quantity $\langle a\rangle$ is non-zero not because strains are persistent butbecause the odds favour net stretching of a surface element by a succession of randomly related short-term strains.

4. Statistical properties of the k^{-1} regime

In treating the higher statistics of the convected scalar field, we shall first examine the case of a white-noise $a(t)$, where clean explicit results are easy to get, and then discuss the deviations expected in more realistic cases. Consider first the statistics of the spatial derivatives of the scalar field $\psi(\mathbf{x}, t)$. A convenient measure of these derivatives is

$$I_n = \int_{\text{blob}} (\partial^n \psi/\partial z^n)^2\, dV, \tag{4.1}$$

where the integration is over the entire volume of a single sheet and z is normal to the sheet. It follows directly from (3.5), and the invariance of the profile shape during convection, that

$$I_n = (\Psi/N)\, b^{-2n} \int_0^\infty x^{2n} f(x)\, dx, \tag{4.2}$$

where N is the number density of sheets. The statistics of I_n therefore are those of b^{-2n}.

In the freely evolving b^{-1} regime (no input after the initial instant), I_n is lognormal, according to (2.31). Its intermittency, as measured by kurtosis, say, increases rapidly with n and with the logarithmic variance $\langle q'^2\rangle$, according to the moment formula

$$\begin{aligned}\langle (b/b_0)^{-n}\rangle = \langle e^{nq}\rangle &= \exp\left[n\langle q\rangle + \tfrac{1}{2}n^2\langle q'^2\rangle\right] \\ &= \exp\left[n\langle a\rangle t + n^2 ct\right].\end{aligned} \tag{4.3}$$

The corresponding formula for the regime with steady input is found from (3.10) to be

$$\langle (b/b_0)^{-n}\rangle = [\exp(n\langle a\rangle t + n^2 ct) - 1]/(n\langle a\rangle t + n^2 ct). \tag{4.4}$$

Equation (3.10) describes a somewhat more intermittent distribution than the lognormal one, since it is an average over lognormal distributions (for b) with different parameters. Thus (4.3) yields

$$\langle b^{-n}\rangle/\langle b^{-1}\rangle^n = \langle b^n\rangle/\langle b\rangle^n = \exp[n(n-1)\, ct], \tag{4.5}$$

while (4.4) gives

$$\langle b^{-n}\rangle/\langle b^{-1}\rangle^n \approx \exp[n(n-1)\, ct]\, (\langle a\rangle t + ct)^n/(n\langle a\rangle t + n^2 ct) \quad (ct \gg 1). \tag{4.6}$$

The ratios given by (4.6) are larger than those given by (4.5), but by a factor small compared with the exponential.

The amplitude difference $\Delta\psi(\mathbf{r}) = \psi(\mathbf{x}+\mathbf{r}) - \psi(\mathbf{x})$ [we suppress the argument $\mathbf{x}$ in $\Delta\psi(\mathbf{r})$] is related to $F(k)$ by

$$[\Delta\psi(\mathbf{r})]^2 = 2\int_0^\infty F(k)\,[1-g(kr)]\,dk, \tag{4.7}$$

where
$$g(kr) = \begin{cases} \sin(kr)/rk \quad \text{(three dimensions)}, \\ (2/\pi)\displaystyle\int_0^1 \cos(krx)\,(1-x^2)^{-\frac{1}{2}}\,dx \quad \text{(two dimensions)}. \end{cases} \tag{4.8}$$

If $F(k)\propto k^{-1}$ is substituted into (4.7), the integral converges at small k and diverges logarithmically at large k, in both three and two dimensions. This shows that (in constrast to a $k^{-\frac{5}{3}}$ range) $\Delta\psi(r)$ in a long k^{-1} range is not a local function in the sense that it is determined by the excitation at wavenumbers $k \sim 1/r$. Instead, it depends on all wavenumbers $k \gtrsim 1/r$, up through the top of the k^{-1} range.

The qualitative statistics of $\Delta\psi(\mathbf{r})$ can be simply inferred when the scalar field consists of little thin sheets. The univariate probability distribution of $\psi(\mathbf{x})$ then has the form

$$P_\psi(\psi) = (1-h)\,\delta(\psi) + hp(\psi), \tag{4.9}$$

where h is the fraction of the total fluid volume occupied by sheets of scalar and $p(\ )$, the probability distribution within a sheet, is assumed to be the same for all sheets. The volume fraction h is constant in time for the freely evolving case described by (2.31) but increases linearly with time in the case of (3.10). Both h and $p(\ )$ are invariant under the convective distortion; $p(\ )$ is the rearrangement of the amplitude profile across the sheets.

If $\mathbf{r}$ is so large that $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ lie in the same sheet with negligible probability, then $\psi(\mathbf{x}+\mathbf{r})$ and $\psi(\mathbf{x})$ are statistically independent, so that, by a general formula for composition of independent variables, the probability distribution $P_\Delta(\Delta\psi)$ is given by

$$P_\Delta(\Delta\psi) = \int_{-\infty}^{\infty} P_\psi(v)\,P_\psi(\Delta\psi+v)\,dv. \tag{4.10}$$

Now suppose that there exists a b^{-1} regime, described by (3.10), which is extensive in the sense $\ln(b_1/b_2) \gg 1$, where b_1 and b_2 mark the limits of b^{-1} behaviour for $P(b)$. Then all but a small fraction of the sheets have thicknesses between b_1 and b_2. It follows that (4.10) is a good approximation to $P_\Delta(\Delta\psi)$ for $r > b_1$.

On the other hand, if r lies well within the b^{-1} range, sheets with $b \gg r$ contribute negligibly to $\Delta\psi(r)$ since, for such sheets, there is a high probability that $\psi(\mathbf{x}+\mathbf{r}) \approx \psi(\mathbf{x})$, if the amplitude profile across the sheets is any reasonable function. But for $b \ll r$, $\psi(\mathbf{r})$ and $\psi(\mathbf{x}+\mathbf{r})$ again almost certainly refer to different sheets and are statistically independent. In this case, it follows that

$$P_\Delta(\Delta\psi) \approx \int_{-\infty}^{\infty} [P_\psi(v)]_r\,[P_\psi(\Delta\psi+v)]_r\,dv, \tag{4.11}$$

where
$$[P_\psi(\psi)]_r = (1-h_r)\,\delta(\psi) + h_r p(\psi) \tag{4.12}$$

and
$$h_r/h \approx [\ln(r/b_2)]/[\ln(b_1/b_2)]. \tag{4.13}$$

Here h_r is the fraction of fluid volume occupied by sheets with $b > r$. Note that no cross-contribution appears in (4.11), with $\mathbf{x}$ in a blob $b < r$ and $\mathbf{x}+\mathbf{r}$ in a blob $b > r$. Such contributions are negligible if r lies well within an extensive b^{-1} range as assumed. The reason is that our underlying assumption of effectively uniform straining over any single sheet implies also uniform straining over the neighbourhood of any sheet. Since the sheets all start out with thickness b_0, this precludes that a sheet with $b \ll r$ could lie within a distance r of a sheet with $b \gg r$.

To illustrate (4.11), take $p(\psi) = \delta(\psi - \psi_0)$; that is, uniform amplitude ψ_0 within the sheets. Then

$$P_\Delta(\Delta\psi) \approx [(1-h_r)^2 + h_r^2]\,\delta(\Delta\psi) + h_r(1-h_r)\,[\delta(\Delta\psi + \psi_0) + \delta(\Delta\psi - \psi_0)]. \quad (4.14)$$

Here the h_r^2 term in the coefficient of $\delta(\Delta\psi)$ gives the probability that both $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ lie in sheets with $b < r$, while the remaining term is the probability that neither $\mathbf{x}$ nor $\mathbf{x}+\mathbf{r}$ lies in such a sheet.

Equation (4.14) and, more generally, (4.11) show that the intermittency of the $\Delta\psi$ distribution exhibits a slow logarithmic increase as r decreases within the b^{-1} range. But as r decreases below b_2, the fraction of sheets with $b < r$ becomes very small, and the intermittency then increases rapidly. The asymptotic b^{-1} range is exactly self-similar in the sense that there are equal numbers of sheets in equal logarithmic steps of b, throughout the range, and the probability distribution of amplitude profiles in the sheets is independent of b. The logarithmic departure of the $\Delta\psi$ distribution from self-similarity is due to the non-local dependence of $\Delta\psi$ on sheet scale size, as indicated already by (4.7).

$P_\Delta(\Delta\psi)$ exhibits qualitatively similar behaviour in the case of the freely evolving b^{-1} range described by (2.31). But here a substantial fraction of the sheets lies outside the b^{-1} range even for large b_1/b_2, as a result of the fact that the logarithmic widths of the range and of the roll-off from it are both measured by $(ct)^{\frac{1}{2}}$. Consequently (2.30) is inaccurate by a factor of order one, for this case.

Now we wish to consider the more realistic case in which the velocity field has finite correlation times, but is still a stationary, homogeneous, isotropic process. If $a(t)$ is sufficiently statistically independent for long time separations, then $q(t)$, as given by (2.29), becomes asymptotically normally distributed for large t (Lumley 1972; Rosenblatt 1972). We shall return to the meaning of 'sufficiently statistically independent'. First, it is important to examine the asymptotic approach to normality for q, and lognormality for b, when it does occur. For this purpose suppose that $a(t)$ and $a(t')$ are completely statistically independent if $t-t'$ exceeds a time Δt. Then, as t increases indefinitely, the fractional deviation of any given moment of $q(t)$ from normal values decreases towards zero. But for any given value of t, however large, moments of $q(t)$ of sufficiently high order are badly approximated by normal values.

The nature of the approximation of b moments by lognormal values is much worse (Novikov 1971). As $t \to \infty$, $\ln\langle b^n\rangle$ approaches lognormal values, based on the values of $\langle b\rangle$ and $\langle b^2\rangle$, with a fractional error which tends to zero. But the fractional deviation of $\langle b^n\rangle$ itself from lognormal values need not decrease as t increases. In particular the ratios $\langle[b(t)]^n\rangle/\langle[b(t')]^n\rangle$, for $t-t' \sim \Delta t$, continue to depend on the detailed $a(t)$ statistics no matter how large t becomes.

These deviations from literal lognormality are associated with the far tails of the $P(b)$ function, at long t. Statistical quantities like $F(k)$ and $P_{\Delta}(\Delta\psi)$ are unaffected for values of k and r which lie within the b^{-1} range. On the other hand, I_n and related functionals of the spatial derivatives of the scalar field are sensitive to the errors of approximation by lognormality.

The statistical independence properties needed for asymptotic normality of q (Lumley 1972; Rosenblatt 1972) require not only that $a(t)$ lose coherence with itself for long time separations, but that it exhibit certain mixing, or ergodic properties. Suppose that the fluid executes a statistically stationary motion such that the trajectories of the scalar sheets are ergodic within subvolumes of the fluid but there is no migration between cells. Then $q(t)$ for any subvolume may become asymptotically normal for large t. But if the $a(t)$ statistics differ from subvolume to subvolume, the asymptotic normal distributions will have different means and variances. The grand distribution over the entire volume will, in general, not be normal. The analysis of the b^{-1} range in §3 remains valid if carried out in the subvolumes and then averaged. Thus (3.11) and (3.12) become

$$P(b) \approx R\langle\bar{a}^{-1}\rangle b^{-1}, \quad F(k) \approx \chi\langle\bar{a}^{-1}\rangle k^{-1}, \tag{4.15}$$

where $\bar{a}$ denotes the long-time average of $a(t)$ for a single blob and angular brackets denote, as before, an average over the entire fluid. Our ergodic assumption implies that $\bar{a}$ equals the ensemble average within a subvolume.

It is not clear to what extent the motion of fluid elements in actual turbulence is ergodic. Both theory and computer experiments suggest that there is trapping of particles in a frozen, random, incompressible, homogeneous velocity field in two dimensions (Kraichnan 1970). However, the same experiments give no evidence of similar trapping in two-dimensional flows which vary randomly in time, or in three-dimensional flows, whether frozen or not. We conjecture that

$$\int_0^t a(s)\,ds$$

is asymptotically normal for stationary, homogeneous, incompressible Navier–Stokes turbulence at all $D \geqslant 2$, provided that the turbulence is maintained by normally distributed, or otherwise sufficiently random, driving forces.

5. The k^{-1} regime with molecular diffusivity

When the molecular diffusivity κ is non-zero, the k^{-1} range extends to

$$k \sim (\langle a\rangle/\kappa)^{\frac{1}{2}}$$

while, for higher wavenumbers, $F(k)$ falls rapidly as k increases (Batchelor 1959). The statistics of our little sheets are more complicated when b^{-1} lies in the latter, diffusion-dominated range because neither the level nor, usually, the shape of the amplitude profile across the sheets is constant in time.

The scalar field obeys

$$(\partial/\partial t - \kappa\nabla^2 + \mathbf{u}\,.\,\nabla)\,\psi = 0. \tag{5.1}$$

In our case of little plane sheets, (5.1) becomes

$$[\partial/\partial t - \kappa\partial^2/\partial z^2 - a(t)\,z\partial/\partial z]\,\psi(z,t) = 0, \tag{5.2}$$

where z is the co-ordinate normal to the sheet in a co-ordinate system whose origin stays centred in the sheet, and which rotates with the sheet, but which does not stretch. Define the function $\phi(zk_0, \tau)$ as follows. The initial profile is

$$\phi(zk_0, 0) = \psi(z, 0), \tag{5.3}$$

where $k_0 = b_0^{-1}$, while

$$(\partial/\partial t - \kappa\partial^2/\partial z^2)\,\phi(zk_0, \kappa k_0^2 t) = 0. \tag{5.4}$$

In other words, ϕ gives the relaxation of the profile that would occur in the absence of convection. Then it may be verified by substitution that (5.2) is satisfied by

$$\psi(z,t) = \phi(zk, \xi), \tag{5.5}$$

where $k(t)$ obeys (2.28), with $k(0) = k_0$, and $\xi(t)$ obeys

$$d\xi/dt = \kappa k^2 = \kappa k_0^2 \exp\left[2\int_0^t a(s)\,ds\right], \tag{5.6}$$

with $\xi(0) = 0$.

The statistics of $\psi(z,t)$ thus are determined by the joint probability distribution of k and ξ. The statistical problem simplifies when $\phi(zk_0, \kappa k_0^2 t)$ is self-preserving, which means an initial profile which either is Gaussian or has the form $\sin(zk_0)$, with enough wavelengths included in the cross-section so that end effects produce negligible smearing of the associated δ-function spectrum.

The choice

$$\phi(zk_0, 0) = \sin(zk_0), \tag{5.7}$$

throughout the volume of the blob, is particularly simple to work with, and we shall use it, despite its artificiality, as a basis for discussing other cases. With this choice, we no longer require that the blobs be thin sheets. Instead, we assume that the initial blobs are many wavelengths large in all directions but still small enough that strains vary negligibly over a blob. Furthermore, we assume that molecular diffusion makes a negligible change in the total volume of a blob during the times of interest

With this choice, (5.5) and (5.6) yield

$$\psi(z,t) = \zeta(t)\sin(kz), \quad d\zeta/dt = -\kappa k^2\zeta, \quad \zeta(0) = 1, \tag{5.8}$$

with $k(t)$ again obeying (2.28), a result also immediately obvious from (5.2). The joint probability distribution $\hat{P}(k,\zeta)$, conditional upon a given history $a(t)$, then satisfies

$$\partial\hat{P}/\partial t + a\partial(k\hat{P})/\partial k - \kappa k^2\partial(\zeta\hat{P})/\partial\zeta = 0. \tag{5.9}$$

The spectrum function $F(k)$ is related to $P(k,\zeta) = \langle\hat{P}(k,\zeta)\rangle$ by

$$F(k) = \tfrac{1}{2}\sigma\int_0^\infty P(k,\zeta)\,\zeta^2 d\zeta, \tag{5.10}$$

where σ is the fraction of the fluid volume that is occupied by one blob. If the equation whose average is (5.10) is inserted in (5.9), the latter yields

$$\partial\hat{F}(k)/\partial t + a\partial[k\hat{F}(k)]/\partial k + 2\kappa k^2\hat{F}(k) = 0, \tag{5.11}$$

where $F = \langle\hat{F}\rangle$ and a partial integration is performed to obtain the final term.

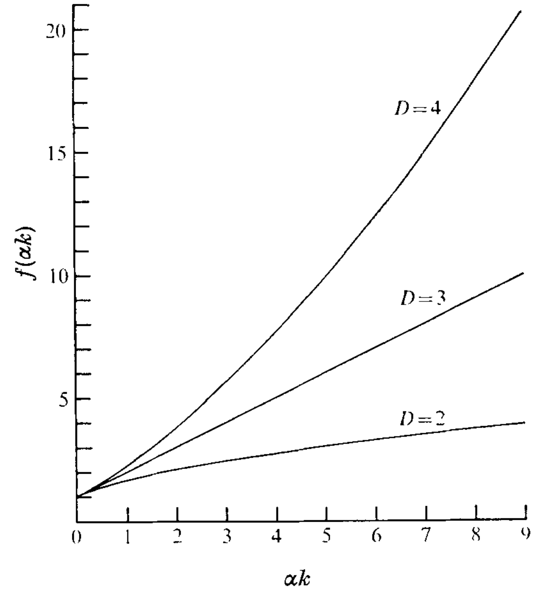

FIGURE 1. The function $f(\alpha k)$ for $D = 2, 3$ and 4.

When $a(t)$ is a white-noise process, the average of (5.11) gives immediately an explicit equation for $F(k)$. To evaluate the term $\langle a\partial[k\hat{F}(k)]/\partial k\rangle$, $\hat{F}(k) - F(k)$ may be expanded as a formal series in powers of $a' = a - \langle a\rangle$, by repeated use of (5.11). Only the term linear in a' gives a surviving contribution in the limit of zero correlation time for $a(t)$. The result is

$$\partial F/\partial t + 2\kappa k^2 F + \langle a\rangle\, \partial(kF)/\partial k - c\partial[k\partial(kF)/\partial k]/\partial k = 0, \tag{5.12}$$

where, as before, $\langle a\rangle$ and c are given by (2.27). For $D = 3$, equation (5.12), with (2.27), is identical, after minor rearrangement, to the spectrum evolution equation obtained by a previous treatment of the white-noise case (Kraichnan 1968).

For a spectral region in a steady state, (5.12) becomes

$$[k\partial^2/\partial k^2 - (D-1)\,\partial/\partial k]\,(kF) = 2\alpha^2 k^2 F, \quad \alpha^2 = D(D+2)\,\kappa/A. \tag{5.13}$$

If $\kappa = 0$, there are two power-law solutions, $F(k) \propto k^{-1}$ and $F(k) \propto k^{D-1}$. The first, of course, represents the k^{-1} range discussed previously. The second is a state of absolute statistical equilibrium in which there is equipartition among all degrees of freedom of the scalar field. If $\kappa > 0$, the solution reducing to (3.12) at small k is

$$F(k) = \chi(\langle a\rangle\, k)^{-1} f(\alpha k\surd 2)\exp(-\alpha k\surd 2), \tag{5.14}$$

where $f(0) = 1$ and $f(x)$ is a solution of Kummer's equation

$$xd^2f/dx^2 - (2x + D - 1)\,df/dx + (D-1)f = 0. \tag{5.15}$$

For $D = 3$, the solution is $f(x) = 1 + x$ (Yee 1968, private communication), and the general solution for odd D is a polynomial of degree $\frac{1}{2}(D-1)$. For $D = 5$, $f(x) = 1 + x + \frac{1}{3}x^2$. For even D, $f(x)$ is transcendental and must be found by numerical integration. It rises monotonically with x and is $O[x^{\frac{1}{2}(D-1)}]$ at large x (Abramowitz & Stegun 1965). Figure 1 shows $f(x)$ for $D = 2$, 3 and 4.

Although we derived (5.12) by taking special input statistics, it is generally valid for any input or initial statistics of the scalar field. This is because the dynamics are linear and moments of different order in the scalar field are decoupled. Of course an input term must be added to (5.12) at wavenumbers where scalar variance is fed into the system. Equation (5.14) is valid in a steady state whatever the statistics of the input at wavenumbers below the k^{-1} range.

According to (5.14), the spectrum fall-off at large k is essentially exponential. On the other hand, if fluctuations in $a(t)$ are neglected [$c = 0$ and $\langle a \rangle$ unchanged in (5.12)], equation (5.14) is replaced by

$$F(k) = \chi(\langle a \rangle k)^{-1} \exp(-\kappa k^2/\langle a \rangle) = \chi(\langle a \rangle k)^{-1} \exp(-\alpha^2 k^2/D). \tag{5.16}$$

This recovers the original result of Batchelor (1959) and gives a Gaussian fall-off at large k. Evidently, the spectrum in the dissipation range is profoundly affected by fluctuations in $a(t)$.

Equation (5.9) leads to the following equation for the full joint probability distribution $P(k, \zeta)$, in analogy with the passage from (5.11) to (5.12):

$$\partial P/\partial t - [k\partial^2/\partial k^2 - (D-1)\,\partial/\partial k]\,(kP) - \alpha^2 k^2 \partial(\zeta P)/\partial\zeta = 0. \tag{5.17}$$

In the k^{-1} regime, the univariate distribution $P(k)$ is unaffected by κ at all k and is given, cf. (3.11), by

$$P(k) = R(\langle a \rangle k)^{-1}, \tag{5.18}$$

where R is the rate of input of blobs. $P(b)$ is normalized by Rt_*, where t_*, the total time of evolution, must be long compared with $\langle a \rangle^{-1} \ln(k/k_0)$ at all wavenumbers k that are of interest. We shall write

$$P(k, \zeta) = P(k)\, P(\zeta|k), \tag{5.19}$$

where $P(\zeta|k)$, the probability distribution of ζ conditional on a sharp value k, is normalized by

$$\int_0^\infty P(\zeta|k)\, d\zeta = 1.$$

Then (5.17) and (5.18) yield the steady-state relation

$$[k\partial^2/\partial k^2 - (D-1)\,\partial/\partial k]\, P(\zeta|k) + \alpha^2 k \partial[\zeta P(\zeta|k)]/\partial\zeta = 0. \tag{5.20}$$

It may be verified by substitution that the solution of (5.20) which obeys the boundary condition $P(\zeta|0) = \delta(\zeta - 1)$ is

$$\zeta P(\zeta|k) = N_D(\alpha k)^D |\ln\zeta|^{-\frac{1}{2}(D+2)} \exp(-\alpha^2 k^2/4|\ln\zeta|) \quad (0 < \zeta < 1), \tag{5.21}$$

where

$$N_D = \begin{cases} 2^{-D}/[\frac{1}{2}(D-2)]! & (\text{even } D), \\ = \frac{1}{2}\pi^{-\frac{1}{2}}[\frac{1}{2}(D-1)]!/(D-1)! & (\text{odd } D). \end{cases}$$

The corresponding distribution $Q(\xi|k)$ for $\xi = -\ln\zeta$ is then

$$Q(\xi|k) = N_D(\alpha k)^D \xi^{-\frac{1}{2}(D+2)} \exp(-\alpha^2 k^2/4\xi) \quad (0 < \xi < \infty). \tag{5.22}$$

For $\alpha k \ll 1$, $P(\zeta|k)$ shows a sharp peak near $\zeta = 1$. This is already substantially broadened for $\alpha k = 1$, and the maximum disappears completely above a moderately greater value of αk, which depends on D. (It is $\alpha k = 2$ for $D = 2$.) The singularity at $\zeta = 0$, which differs only logarithmically from a ζ^{-1} singularity, grows more prominent as αk increases. For $\alpha k \gg 1$, $P(\zeta|k)$ descends smoothly from the singularity to its eventual cut-off, associated with the exponential factor in (5.21). $Q(\xi|k)$ is a notably skewed distribution exhibiting very high intermittency. Its moments of order $\frac{1}{2}D$ and higher do not exist.

Equation (5.22) can be rewritten as

$$Q(\xi|k) = \int_0^\infty \delta(\xi - \tfrac{1}{2}\kappa k^2/\langle a\rangle v)\, Z(v)\, dv, \tag{5.23}$$

where
$$Z(v) = N_D(2D)^{\frac{1}{2}D} v^{\frac{1}{2}(D-2)} \exp(-\tfrac{1}{2}Dv), \quad \int_0^\infty Z(v)\,dv = 1. \tag{5.24}$$

This has the following interpretation. A time-invariant statistically sharp stretching function $\langle a\rangle$ would result in the distribution

$$Q(\xi|k) = \delta(\xi - \tfrac{1}{2}\kappa k^2/\langle a\rangle), \tag{5.25}$$

corresponding to (5.16). Thus (5.23) and (5.24) represent the actual $Q(\xi|k)$ as the result of a statistical distribution $Z(v)$ of fictitious constant stretchings $v\langle a\rangle$. Equation (5.24) gives
$$\int_0^\infty Z(v)\, v\, dv = 1,$$
so that the mean of the fictitious stretchings equals the actual mean $\langle a\rangle$.

The singularity in $P(\zeta|k)$ at $\zeta = 0$ is associated with the behaviour of $Z(v)$ in the neighbourhood of $v = 0$; that is, with very small effective stretchings which permit high attenuation for a given growth of k from its initial value k_0. In the actual physics, where $a(t)$ fluctuates in time, this corresponds not only to slow monotonic growth of k but also to realizations in which k attains high values, where ζ is dissipated rapidly, and then *decreases*, when $a(t)$ becomes negative for an interval of time.

The moments
$$M_n(k) = \int_0^\infty \zeta^n P(\zeta|k)\, d\zeta \tag{5.26}$$
obey
$$\partial M_n/\partial t - [k^2\partial^2/\partial k^2 - (D-1)\, k\partial/\partial k] M_n + n\alpha^2 k^2 M_n = 0, \tag{5.27}$$
an equation which follows from taking moments of (5.17) and performing a partial integration on the $\partial/\partial\zeta$ term. Comparison with (5.13) shows that (5.27) has the steady-state solution

$$M_n(k) = f(\alpha k n^{\frac{1}{2}}) \exp(-\alpha k n^{\frac{1}{2}}). \tag{5.28}$$

The preceding discussion shows that $Q(\xi|k)$ has no very spectacular dependence on D. It is of interest, however, to examine the behaviour as $D \to \infty$. $Z(v)$, given by (5.24), has its maximum at $v = 1-(2D)^{-1}$. As $D \to \infty$, the peak gets sharper, and the limiting form is $Z(v) = \delta(v-1)$. Thus (5.16) and (5.25) are exact in the limit of infinite dimension. This is also clear from (2.27), which gives $c/\langle a\rangle \to 0$ in the limit.

Much of the analysis for the special case of blobs with sinusoidal profiles applies also to the general sheet profile, described by (5.5), (5.6) and (2.28). We have already noted the general validity of the spectrum equation (5.12). Moreover the parameter ξ, which equals $-\ln \zeta$ in the special case, obeys the same equation of motion (5.6) and initial condition $\xi(0) = 0$ in the general case. Thus the probability distribution $Q(\xi|k)$ is still given by (5.22) in the general case.

6. Statistics of $\psi(\mathbf{x}, t)$ and its spatial derivatives

The explicit solution for the joint probability distribution $P(k)\,Q(\xi|k)$ found in §5, for the case of isotropic turbulence with rapid time variation, permits the determination of a great variety of probability distributions and moments for the scalar amplitude $\psi(\mathbf{x}, t)$, and its spatial derivatives, by appropriate quadratures. We shall indicate the general technique for doing this, and then give a few particular results.

It is first necessary to point out the relationship between the single-sheet distribution functions and the distribution functions of the fields $\psi(\mathbf{x}, t)$, $\partial\psi(\mathbf{x}, t)/\partial x_1$, etc. Let w be any linear functional of $\psi(\mathbf{x})$; in particular w can stand for $\psi(\mathbf{x})$ itself or any of its spatial derivatives. Let $P_w(w|k_0, \tau)$, where $\tau = \kappa k_0^2 t$, be the probability distribution for w for a single, randomly oriented and positioned sheet which suffers diffusion but not convection. Thus $P_w(w|k_0, \tau)$ is a functional of the function ϕ defined by (5.3) and (5.4) and of the characteristic sheet volume $A_0/k_0 = V_0$, where A_0 is the $(D-1)$-dimensional area of the input sheet. If an initial amplitude distribution is taken, by altering (5.3) to

$$\psi(z, 0) = \beta\phi(zk_0, 0),$$

where β is a random variable of prescribed distribution, these statistics are also included in $P_w(w|k_0, \tau)$. The normalization

$$\int_{-\infty}^{\infty} P_w(w|k_0, \tau)\, dw = 1 \tag{6.1}$$

is used.

Under the assumption that the volume of a single sheet is small compared with the total volume of the flow, $P_w(w|k_0, \tau)$ can be accurately approximated in the form

$$P_w(w|k_0, \tau) = [1-\sigma(\tau)]\,\delta(w) + \sigma(\tau)\, P'_w(w|k_0, \tau). \tag{6.2}$$

Here $\sigma(\tau)$ is the effective fraction *of the total flow volume* in which w is non-negligible and $P'_w(w|k_0, \tau)$ is the probability distribution of w, within that volume fraction, again normalized to unity. Obviously $\sigma(\tau)$ is proportional to V_0, and it is, in general, an increasing function of τ, since diffusion tends to spread the excitation in space.

It follows directly from (5.5) that the corresponding single-sheet probability distributions in the presence of convection are $P_w(w|k, \xi)$ and $P'_w(w|k, \xi)$, while (6.2) becomes

$$P_w(w|k, \xi) = [1-\sigma(\xi)]\,\delta(w) + \sigma(\xi)\, P'_w(w|k, \xi). \tag{6.3}$$

Note that convection affects the volume fraction σ only indirectly, through ξ, since the convection preserves volumes.

The total field $w(x)$ is a sum of contributions from all the sheets, since w is a linear functional of ψ. Consequently, the overall distribution $P_w(w)$ is the convolution of the individual distributions $P_w(w|k,\xi)$ of all the sheets, provided that the sheets are statistically independent of each other. It follows that the characteristic functions

$$C_w(s) = \int_{-\infty}^{\infty} P_w(w)\, e^{iws}\, dw, \quad C_w(s|k,\xi) = \int_{-\infty}^{\infty} P_w(w|k,\xi)\, e^{iws}\, dw$$

are related by

$$C_w(s) = \prod_{n=1}^{N} C_w(s|k_n,\xi_n), \tag{6.4}$$

where k_n and ξ_n are the parameters of the nth sheet. In the limit of a large number N of sheets, (6.3) becomes

$$\ln C_w(s) = \int_0^\infty d\xi \int_0^\infty dk \ln\left[C_w(s|k,\xi)\right] P(k)\, Q(\xi|k), \tag{6.5}$$

where, according to the definitions in §5, $P(k)$ is the distribution of k, normalized to N, and $Q(\xi|k)$ is the distribution of ξ at given k, normalized to unity.

Two approximations, whose validity must be examined carefully, lead to successive simplifications of (6.5). The Fourier transform of (6.3) is

$$C_w(s|k,\xi) = 1 - \sigma(\xi) + \sigma(\xi)\, C'_w(s|k,\xi). \tag{6.6}$$

Thus if $\sigma(\xi) \ll 1$, as we have assumed, this suggests that (6.5) can be replaced by

$$\ln C_w(s) = \int_0^\infty d\xi \int_0^\infty dk\left[-\sigma(\xi) + \sigma(\xi)\, C'_w(s|k,\xi)\right] P(k)\, Q(\xi|k). \tag{6.7}$$

If, in addition,

$$N\int_0^\infty d\xi \int_0^\infty dk\, \sigma(\xi)\, Q(\xi|k)\, P(k) \ll 1,$$

which means that all N sheets occupy only a small fraction of the fluid, it is further suggested that $\ln C_w(s) \ll 1$, so that $C_w(s) \approx 1 + \ln C_w(s)$. Then (6.7) can be transformed back to yield, finally,

$$P_w(w) = \left[1 - \int_0^\infty d\xi \int_0^\infty dk\, \sigma(\xi)\, P(k)\, Q(\xi|k)\right]\delta(w) + \int_0^\infty d\xi \int_0^\infty dk\, \sigma(\xi)\, P'_w(w|k,\xi)\, P(k)\, Q(\xi|k). \tag{6.8}$$

Equation (6.8) clearly describes the case of non-overlapping sheets, a condition which is consistent with statistical independence of the sheets only if the sheets are sparse. The conditions for the accurate validity of (6.8) are realizable for the transient convection at $\kappa = 0$ analysed in §§3 and 4, provided that the total number of sheets initially present, or the total input Rt, is appropriately small. But the steady-state distributions considered in §5 violate the conditions because, no matter how small R is, the total number of sheets grows linearly with t and is infinite in the steady state. Moreover, $\sigma(\zeta)$ grows with ξ for most choices of initial profile (the sinusoidal profile (5.7) is an exception). However, the total number of sheets with $k < \alpha^{-1}$ in the steady state is $\sim |\ln(\alpha k_0)| R/\langle a\rangle$, which can be as small as desired. The violation of the conditions for (6.8) therefore is associated with sheets of very high k and consequently very low amplitude which fill the flow volume with a background of low-level excitation. We therefore anticipate that, if (6.8) is applied to the steady state, $P_w(w)$ may exhibit a non-integrable singularity at zero amplitude w, which would disappear if (6.5) were used instead. The situation is analogous to multiple soft-photon divergencies in radiation theory.

In practice, (6.5) leads to awkward numerical evaluations in cases where (6.8) does not. We therefore find it both easier and more illuminating to use (6.8) even where it should not be used and then identify the resulting misbehaviour at $w = 0$ and indicate qualitatively how it would change if (6.5) were used.

Now we shall examine the particular case where the input sheets have the sinusoidal profile (5.7) and w is taken as $\partial^n\psi(\mathbf{x})/\partial x_1^n$, where x_1 is any space direction and $n = 0$ denotes simply $\psi(\mathbf{x})$. For this case, the initial (input) distributions have the form

$$P'_n(w|k_0, 0) = k_0^{-n}\, Y_n(k_0^{-n}w), \tag{6.9}$$

where $Y_n(w)$ is the distribution of $\partial^n\psi(\mathbf{x})/\partial x_1^n$ associated with randomly oriented sheets of profile $\beta\sin z$ and whatever β statistics are desired. (Note that

$$\partial/\partial x_1 = n_1\partial/\partial z,$$

where $\mathbf{n}$ is a random unit vector.) Then, by (5.5) and (5.8), $\sigma(\xi) = \sigma(0) \equiv \sigma$, while

$$P'_n(w|k,\xi) = k^{-n}e^{\xi}Y_n(e^{\xi}k^{-n}w). \tag{6.10}$$

This expression, used in (6.8) together with formulae for $P(k)\,Q(\xi|k)$ from §3 or §5, gives an explicit result for the distribution $P_n(w)$ of $\partial^n\psi(\mathbf{x})/\partial x_1^n$ in terms of an integral over the input distribution $Y_n(w)$.

Moments of the several distributions may be defined by

$$\left.\begin{aligned} M_n^r = \int_{-\infty}^{\infty} w^r P_n(w)\,dw, \quad M_n^r(k,\xi) = \int_{-\infty}^{\infty} w^r P'_n(w|k,\xi)\,dw,\\ M_n^r(0) = \int_{-\infty}^{\infty} w^r Y_n(w)\,dw. \end{aligned}\right\} \tag{6.11}$$

Then (6.8) gives

$$M_n^r = \int_0^\infty d\xi \int_0^\infty dk\,\sigma(\xi)\,M_n^r(k,\xi)\,P(k)\,Q(\xi|k), \tag{6.12}$$

while (6.10) yields

$$M_n^r(k,\xi) = e^{-r\xi}k^{nr}M_n^r(0). \tag{6.13}$$

Also, for the sinusoidal sheets

$$M_n^r(0) = M_0^r(0)\,\langle n_1^{nr}\rangle, \tag{6.14}$$

where
$$\langle n_1^s\rangle = \begin{cases} [1.3\ldots(s-1)]/[D(D+2)\ldots(D+s-2)] & (\text{even } s),\\ 0 \quad (\text{odd } s). \end{cases} \tag{6.15}$$

For the transient regime at $\kappa = 0$ with steady input rate R, $P(k) = k^{-1}Q(q)$ with $q = \ln(k/k_0)$ and $Q(q)$ given by (3.10), while $Q(\xi|k) = \delta(\xi)$. Then (6.12), (6.13) and (4.4) give

$$M_n^r = \sigma M_n^r(0)\,k_0^s[\exp(s\langle a\rangle t + s^2ct) - 1]/(s\langle a\rangle t + s^2ct) \quad (s = nr). \tag{6.16}$$

The rapid rise of M_n^r with nr confirms the essentially lognormal behaviour of $\partial^n\psi(\mathbf{x})/\partial z^n$ found in §4.

For the steady-state finite κ regime of §5, $P(k)$ is given by (5.18) for k well above k_0, and $Q(\xi|k)$ is given by (5.22). In order to be able to do the integrals in (6.12) analytically, we shall assume (5.18) and (5.22) for *all* k, and then identify the errors produced thereby in the results. With this procedure, the k integration in (6.12) yields

$$M_n^r/M_n^r(0) = \sigma R\langle a\rangle^{-1}\alpha^{-rn}N_D N'_{D+rn}\int_0^\infty \xi^{\frac{1}{2}rn}e^{-r\xi}\xi^{-1}d\xi, \tag{6.17}$$

where

$$N'_s = \begin{Bmatrix} 2^s[\frac{1}{2}(s-1)]! & (\text{odd } s), \\ \pi^{\frac{1}{2}}s!/(\frac{1}{2}s)! & (\text{even } s). \end{Bmatrix} \tag{6.18}$$

For $n = 0$, the integral in (6.17) diverges logarithmically at $\xi = 0$ for all r and diverges at $\xi = \infty$ for $r \leqslant 0$. The ratio of the peak amplitude in a sheet to its initial peak amplitude is $\zeta = e^{-\xi}$. The divergence at $\xi = 0$ therefore is associated with the impermissible extension of the k integration below k_0, since it is at low wavenumbers that amplitudes are unaltered. The behaviour at $\xi = \infty$, or $\zeta = 0$, suggests that $P_0(w)$ behaves like w^{-1}, apart from possible logarithmic factors, at $w = 0$. These inferences are supported by looking directly at the distribution of ζ. Integration of (5.19), with $P(\zeta|k)$ given by (5.21), yields

$$\int_0^\infty P(k,\zeta)\,dk = R\langle a\rangle^{-1}N_D N'_D(\zeta|\ln\zeta|)^{-1}. \tag{6.19}$$

Here the divergence at $\zeta = 1$ is easily seen to come from $k = 0$. The singularity in (6.19) at $\zeta = 0$ is non-integrable. Similar behaviour for $P_0(w)$, as given by (6.8), can be verified at $w = 0$. The divergence is due to the background of low amplitude, high k sheets, discussed above, and is associated with the breakdown of the approximations leading from (6.5) to (6.8).

If the k integrals were properly cut off at the lower limit k_0, and (6.5) were used, the result would be a function $P_0(w)$ with a tail that behaves like $(w|\ln w|)^{-1}$ down to a value $w(\sigma)$, such that

$$\int_{w(\sigma)}^\infty P_0(w)\,dw \sim 1,$$

and changes behaviour for smaller w so as to be integrable at $w = 0$. The crossover $w(\sigma)$ decreases rapidly as σ decreases, and for $\sigma \ll 1$, the $(w|\ln w|)^{-1}$ tail extends to very small w.

For $n \geqslant 1$ and $r = 0$, the integral in (6.17) exhibits the same divergences at $\xi = 0$ and $\xi = \infty$ as for $n = 0$. This implies a tail to $P_n(w)$ like the tail of $P_0(w)$ at small w, but with logarithmic factors that may depend on n. In contrast to the case $n = 0$, the integrals in (6.17) converge for $n \geqslant 1$, $r > 0$, and (6.17) gives M_n^r accurately if $\sigma \ll 1$. With the integration performed, (6.17) gives

$$M_n^r/M_n^r(0) = \sigma R\langle a\rangle^{-1}\alpha^{-rn}N_D N'_{D+rn}r^{-\frac{1}{2}rn}\Gamma(\tfrac{1}{2}rn). \tag{6.20}$$

When rn is large, (6.20) gives

$$M_n^r \sim \bar{\sigma}\alpha^{-rn}e^{-rn}r^{\frac{1}{2}rn}n^{rn}M_n^r(0), \tag{6.21}$$

with
$$\bar{\sigma} = \sigma R/\langle a\rangle,$$
and
$$M_n^{2r}/(M_n^2)^r \sim \bar{\sigma}^{1-r}\, r^{rn} M_n^{2r}(0)/[M_n^2(0)]^r, \tag{6.22}$$
where Stirling's approximation is used and only factors of exponential strength in r or n are retained. In contrast, (6.16) yields
$$M_n^{2r}/(M_n^2)^r \sim \bar{\sigma}^{1-r} \exp\left[4n^2(r^2-r)\,ct\right] M_n^{2r}(0)/[M_n^2(0)]^r. \tag{6.23}$$
Equation (6.22) shows a markedly less rapid rise of normalized moment values with r and n than the essentially lognormal behaviour exhibited in (6.23). In the steady-state, non-zero κ regime, dissipation cuts off the excitation in very small scales much more rapidly than in the very diffuse tail of the freely growing k^{-1} regime at zero κ. Clearly this has a big effect on high-order moments of high-order derivatives. The rate of growth with r shown in (6.22) is more nearly typical of a normal distribution than a lognormal one. The precise form of the distribution cannot be deduced from these crude asymptotic estimates of course; it must be found, if desired, from (6.8) or (6.5).

Sheets with sinusoidal profiles give the advantage of analytical simplicity and clarity, but they are quite unrealistic. A more physically relevant choice is the Gaussian profile, for which the evolution under pure diffusion is
$$\phi(zk_0, \tau) = (1+2\tau)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}z^2k_0^2/(1+2\tau)\right] \quad (\tau = \kappa k_0^2 t). \tag{6.24}$$
The Gaussian profile is self-preserving under both diffusion and convection, and arbitrary profiles tend towards the Gaussian form after sufficiently long periods of diffusion.

The volume occupied by a sheet of profile (6.24) and area A_0 is not precisely defined. For our purposes it suffices to take
$$\sigma(\tau) = \mu\sigma_0(1+2\tau)^{\frac{1}{2}}, \tag{6.25}$$
where σ_0 is A_0/k_0 divided by the total flow volume, and μ is a small integer such that $\exp(-\frac{1}{2}z^2)$ and all its derivatives of interest are negligible for $z > \mu$. For the Gaussian sheets, (6.10) is replaced by
$$P_n'(w|k,\xi) = k^{-n}(1+2\xi)^{\frac{1}{2}(n+1)}\, Y_n[(1+2\xi)^{\frac{1}{2}(n+1)} k^{-n} w], \tag{6.26}$$
where $Y_n(w)$ now is the distribution, in the centred volume μ, of $\partial^n\psi(\mathbf{x})/\partial x_1^n$ for a randomly oriented sheet of unit area and profile $\beta\exp(-\frac{1}{2}z^2)$, with whatever β statistics are desired. The moment relations (6.13) and (6.14) become
$$\sigma(\xi)\, M_n^r(k,\xi) = \sigma_0(1+2\xi)^{\frac{1}{2}[1-(n+1)r]} k^{nr} M_n^r(0) \tag{6.27}$$
and
$$M_n^r(0) = C(n,r)\, n_1^{nr} M_0^r(0), \tag{6.28}$$
where
$$C(n,r) = \int_{-\infty}^{\infty} [d^n \exp(-\tfrac{1}{2}z^2)/dz^n]^r\, dz \Big/ \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}rz^2)\, dz. \tag{6.29}$$
Particular values of $C(n,r)$ are
$$C(n,2) = 2^{-2n} 2n!/n!, \quad C(1,2r) = 2^{-2r} r^{-r} 2r!/r!. \tag{6.30}$$
If the manipulations which led to (6.17) are carried out for the Gaussian sheets, the result is
$$M_n^r/M_n^r(0) = \sigma_0 R\langle a\rangle^{-1} \alpha^{-rn} N_D N'_{D+rn} \int_0^\infty (1+2\xi)^{\frac{1}{2}[1-(n+1)r]}\, \xi^{-1+\frac{1}{2}rn}\, d\xi. \tag{6.31}$$

The integral in (6.31) behaves like that in (6.17) at $\xi = 0$, with the same interpretation. However, the behaviour at $\xi = \infty$ is different. In (6.31), the integral converges at $\xi = \infty$ for $r > 1$, independently of n. This implies that $P_n(w)$ has a tail which goes like w^{-2} at small w, apart from logarithmic factors, in contrast to the w^{-1} behaviour for sheets with sinusoidal profiles. In accord with the previous discussion, the w^{-2} behaviour is modified to an integrable dependence for w less than some cross-over $w(\sigma_0)$, which goes to zero with σ_0. Again, the modification represents the difference between (6.5) and its approximation (6.8).

For $n \geqslant 1$ and $r > 1$, the integral in (6.31) converges at both limits and M_n^r is given accurately if $\sigma_0 \ll 1$.† For large r and n, the factors of exponential strength in the evaluation of the integral are $n^{-\frac{1}{2}r}\, 2^{-\frac{1}{2}nr}$. Consequently, the asymptotic behaviour of M_n^r, corresponding to (6.21) and (6.22), is

$$M_n^r \sim \overline{\sigma}_0 \alpha^{-rn} e^{-\frac{1}{2}rn} (rn)^{\frac{1}{2}rn} n^{-\frac{1}{2}r} M_n^r(0), \tag{6.32}$$

$$M_n^{2r}/(M_n^2)^r \sim \overline{\sigma}_0^{1-r} r^{rn} M_n^{2r}(0)/[M_n^2(0)]^r, \tag{6.33}$$

with $\overline{\sigma}_0 = \sigma_0 R/\langle a \rangle$. Comparison of (6.22) and (6.33) shows that the increase in intermittency of the spatial derivatives under the combined action of convection and diffusion is essentially the same for sheets with Gaussian profiles and for sheets with sinusoidally oscillating profiles.

The spectrum (5.14) for the steady-state regime is related to the derivative moments by

$$M_n^2 = \int_0^\infty k^{2n} F(k)\, dk. \tag{6.34}$$

Since $F(k)$ is universal and independent of the sheet profile, (6.34) provides a consistency check on the moment asymptotics. If only factors exponential in n are retained, (6.34) gives

$$M_n^2/M_1^2 \sim (\tfrac{1}{2}\alpha^2 e^2)^{1-n} n^{2n}, \tag{6.35}$$

for all finite D. The f factor in F does not affect the exponential dependence. Equation (6.35) is recovered both from (6.21), with the aid of (6.14), and from (6.32), with the aid of (6.28) and (6.30), so that the asymptotic analysis agrees with (6.34) for both of the profiles considered.

7. Summary and discussion

Our analysis started with the stretching of line elements, local wave vectors and surface elements by isotropic turbulence in a D-dimensional space. The first principal result was that the same stochastic stretching parameter $a(t)$ controls the logarithmic rate of growth of all three of these different geometric objects if the velocity field has reversal invariance. For a white-noise velocity field (one driven such that correlation times are much shorter than eddy circulation times), the process $a(t)$ was constructed explicitly, and the mean $\langle a \rangle$ and covariance integral $c(t)$ expressed in terms of the covariance of the velocity field, by

† The substitution $x = 1 + 2\xi$ transforms the integral to an elementary, analytically integrable form, if r is even.

(2.27). The induced statistics for the stretched line elements, or $(D-1)$-dimensional surface elements, were found to be lognormal. We found $\langle a\rangle/c = D$, so that fluctuations in the stretching process become negligible for infinite D. In the more realistic case of velocity correlation times comparable with eddy circulation times, we found that statistics of the stretched line elements are asymptotically lognormal, subject to certain necessary conditions and restrictions on interpretation (§§3 and 4).

The results on stretching of line elements and surface elements are directly applicable to the statistics of the k^{-1} spectral regime (Batchelor 1959) resulting from convection of a sparse distribution of little sheets of passive scalar whose dimensions are small compared with the correlation scales of the shearing field. We examined two cases in §§3 and 4: the first where all the sheets are put in at $t = 0$ and the second where sheets are added at a constant rate R. For both cases a number of exact statistical results were obtained under the assumption of rapid velocity-field variation. In the first case, the spatial derivatives of the scalar field were found to have lognormal statistics, while the second case yielded statistics slightly more intermittent than lognormal. These results are given in (4.1)–(4.6).

The statistics of the difference field $\psi(\mathbf{x}+\mathbf{r})-\psi(\mathbf{x})$ were found to vary only very slowly with r for r^{-1} in the k^{-1} spectral range. The intermittency of the distribution of this quantity increases logarithmically with decreasing r [(4.11)–(4.14)].

In §§5 and 6, the analysis was extended to sheets of scalar undergoing simultaneous convection and molecular diffusion. The key to the extension is the similarity solution (5.5) which relates the evolution of sheets under combined convection and diffusion to their evolution under diffusion alone. Here ξ is the non-dimensionalized effective time over which diffusion has acted and k is the characteristic wavenumber that the sheet would have under pure convection.

With the aid of a special choice of sheet profile (sinusoidal oscillation) we derived the spectrum evolution equation (5.12) and found its steady-state solution (5.14) over the k^{-1} and dissipation ranges. These results are independent of input scalar-field statistics (because the dynamics are linear), and they generalize to D dimensions results found previously (Kraichnan 1968) for $D = 3$, under the assumption of a rapidly varying velocity field. For all finite D, the spectrum falls off essentially exponentially in the scalar dissipation range. This behaviour is due to the fluctuations in the effective stretching process $a(t)$ and contrasts with the Gaussian fall-off found by Batchelor (1959) for straining constant in time. We found that the spectrum and indeed, the full joint probability distribution of k and ξ resulting from the actual $a(t)$ can be represented as an average over a distribution $Z(v)$, with $\langle v\rangle = 1$, of constant stretchings $v\langle a\rangle$ [(5.23)–(5.25)]. In the limit $D\rightarrow\infty$, fluctuations in $a(t)$ are negligible, and Batchelor's original result is recovered.

In addition to the spectral information, we obtained an analytical solution for the steady-state joint probability distribution of k and ξ, together with the associated distribution for $\zeta = e^{-\xi}$ and expressions for the moments of the latter [(5.18)–(5.28)]. In the special case of sheets with sinusoidally oscillating profiles,

ζ is the ratio of the sheet amplitude to its initial amplitude. The conditional probability distribution $Q(\xi|k)$ for $\xi = -\ln\zeta$ at given k is highly skewed, and it falls off like $\xi^{-\frac{1}{2}(D+2)}$ at large ξ, so that most of its moments do not exist.

The joint probability distribution for k and ξ, which was obtained, again, for a rapidly varying velocity field, permits the calculation of a wide variety of statistics both for individual sheets and for the scalar field, and its spatial derivatives, resulting from contributions by all the sheets. We found in §6 that there were profound differences between the probability distributions of the spatial derivatives in the cases of the freely growing k^{-1} regime with zero molecular diffusivity and of the steady-state k^{-1} regime (with associated dissipation range) with non-zero diffusivity. The distributions for a sparse collection of sheets in the non-zero diffusivity case displayed low amplitude tails which gave non-integrable singularities at zero amplitude w. The behaviour was found to be approximately like w^{-1} for sheets with sinusoidally oscillating profiles and like w^{-2} for the more realistic case of sheets with Gaussian amplitude profiles. These singularities, which are analogous to soft-photon divergencies in radiation theory, are altered to integrable form, at very small w, when proper account is taken of the overlap of highly diffused sheets [(6.1)–(6.8) and associated discussion].

In addition to peculiar behaviour at small amplitudes, the distributions of spatial derivatives in the regime with non-zero diffusivity exhibit much lower intermittency, and a lower rise of intermittency with the order of the derivative, than the essentially lognormal intermittency found for the zero-diffusivity k^{-1} regime. These results are exhibited by the asymptotic moment expressions (6.23), (6.22) and (6.33). The sinusoidal and Gaussian sheet profiles were found to give essentially the same growth of intermittency with the order of the derivative. It is of interest that the asymptotic moment expressions (6.21) for sinusoidal sheets can be shown to be the geometric mean of two artificial sets of moment values. In the first, all sheets of given k have the same amplitude, and the latter is chosen to give the correct spectrum (5.14). In the second, all sheets of all k have their undiminished input amplitude, but sheets are annihilated selectively at different k's so as, again, to give (5.14). It is also of interest that the same rise of intermittency with derivative order n, shown in (6.22) and (6.33), occurs if the input sheets are fed directly into the dissipation range and there is no k^{-1} range at all [$\alpha k_0 > 1$, where k_0 is the characteristic wavenumber of the input sheets and α is defined in (5.13)]. This intermittency is intrinsic to the rapidly falling dissipation-range spectrum (Kraichnan 1967).

It should be pointed out that, although we expect the qualitative features of the statistics found for the non-zero diffusivity regime with rapidly varying velocity field to hold also for velocity fields with correlation times of the order of eddy circulation times, even the spectrum form (5.14) will show some changes. In the exponentially falling range, the spectrum changes substantially with each doubling of k. But for a sheet with typical history, this doubling takes place in a time of order $\langle a\rangle^{-1}$. If this is also the correlation time of $a(t)$, then clearly the spectrum shape, and higher statistics as well, depend on details of the velocity statistics. There is, then, no universal steady-state dissipation-range spectrum.

Even with given velocity statistics, the small-scale statistics of the scalar field cannot be independent of scalar input statistics. The dynamics are linear, moments of different order in the scalar field are uncoupled and consequently the effect of initial moment ratios is felt forever. Moreover, for zero diffusivity κ the univariate distribution of ψ is invariant under convection. The question arises of how much loss of generality in the description of small-scale statistics is incurred by our choice of sparsely distributed little sheets as the input condition.

If the diffusivity is small, the straining action should eventually draw out scalar blobs of whatever initial form into thin ribbons. If the velocity field has a finite coherence time, a state should then be reached which well approximates our initial state of thin sheets statistically independent of the velocity field. What are left then are the questions of the density of distribution of the sheets and of the difference between little sheets which are statistically independent of each other (our choice) and little sheets which join edge-to-edge to form big ribbons. It seems likely that both questions involve fairly subtle differences in higher statistics which are not crucial at the present level of treatment, although this surely cannot be asserted. It should be pointed out that our analysis runs into trouble if the sheets are densely distributed even if they are initially statistically independent. Overlapping contributions from different sheets can be handled self-consistently by (6.5), but there is illogicality in this because it is unwarranted to assume that overlapping sheets that start out close together (within one velocity-field or shear correlation length) remain statistically independent. Thus we cannot legitimately treat multivariate Gaussian initial statistics by applying our analysis to a dense array of initially independent, overlapping little sheets.

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